Exercise sheet 4

The Optional stopping theorem

Homework:
Please hand in the solutions of Exercises from 4.1 to 4.3 in the lecture on November 27th.

Exercise 4.1 (7 Points)

Consider two stopping times σ and τ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)$. Our aim is to prove that

$$\mathbb{E}[\mathbb{E}[\,\cdot\,|\mathcal{F}_\sigma]|\mathcal{F}_\tau] = \mathbb{E}[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}] = \mathbb{E}[\mathbb{E}[\,\cdot\,|\mathcal{F}_\tau]|\mathcal{F}_\sigma] \qquad P-a.s., \tag{1}$$

i.e., the operators $\mathbb{E}[\,\cdot\,|\mathcal{F}_\sigma]$ and $\mathbb{E}[\,\cdot\,|\mathcal{F}_\tau]$ commute and their composition equals $\mathbb{E}[\,\cdot\,|\mathcal{F}_{\sigma\wedge\tau}]$.

Note: For arbitrary sub-σ-algebras $\mathcal{G}$, $\mathcal{G}' \subset \mathcal{F}$, the conditional expectations $\mathbb{E}[\mathbb{E}[\,\cdot\,|\mathcal{G}]|\mathcal{G}']$, $\mathbb{E}[\mathbb{E}[\,\cdot\,|\mathcal{G}']|\mathcal{G}]$ and $\mathbb{E}[\,\cdot\,|\mathcal{G}\cap\mathcal{G}']$ do **not** coincide in general!

(i) Show that if $\sigma \leq \tau$, then $\mathcal{F}_\sigma \subseteq \mathcal{F}_\tau$, that in general $\sigma \wedge \tau$ is a stopping time and that $\mathcal{F}_{\sigma\wedge\tau} = \mathcal{F}_\sigma \cap \mathcal{F}_\tau$.

(ii) Show that $\{\sigma < \tau\} \in \mathcal{F}_{\sigma\wedge\tau}$ and $\{\sigma \leq \tau\} \in \mathcal{F}_{\sigma\wedge\tau}$.

Hint: For the first assertion, use the fact that $a < b$ if and only if there is a rational $q \in \mathbb{Q}$ such that $a \leq q < b$.

(iii) Conclude that if $A \in \mathcal{F}_\sigma$, then $A \cap \{\sigma \leq \tau\}$ and $A \cap \{\sigma < \tau\}$ belong to $\mathcal{F}_\tau$.

(iv) Show that $\mathbb{E}[Y|\mathcal{F}_\tau]$ is $\mathcal{F}_{\sigma\wedge\tau}$-measurable if Y is an integrable, $\mathcal{F}_\sigma$-measurable random variable. Conclude (1).

Exercise 4.2 (6 Points)

Let $(B_t)_{t\geq 0}$ be a standard Brownian Motion and let $T := \inf\{t \geq 0 : B_t = at - b\}$ for some positive constants $a, b > 0$. Calculate $E[T]$.

Hint: You might want to use Wald's lemma...

Exercise 4.3 (7 Points)
Let $(B_t)_{t\geq 0}$ be Brownian motion.

(i) Use the optional stopping theorem for a martingale that you know to show that, with $\tau_a = \inf\{t \geq 0 : B_t = a\}$,

$$E_0[e^{-\lambda\tau_a}] = e^{-a\sqrt{2\lambda}}, \quad \text{for all } \lambda, a > 0.$$

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(ii) Show that, with $\tau_{-a} = \inf\{t \geq 0 : B_t = -a\}$, we have

$$E_0[e^{-\lambda\tau_a}] = E_0[e^{-\lambda\tau_a} \mathrm{I}_{\{\tau_a < \tau_{-a}\}}] + E_0[e^{-\lambda\tau_{-a}} \mathrm{I}_{\{\tau_{-a} < \tau_a\}}]e^{-2a\sqrt{2\lambda}}.$$

(iii) Use the previous results to show that $\tau = \tau_a \wedge \tau_{-a}$ satisfies

$$E_0[e^{-\lambda\tau}] = \mathrm{sech}\left(a\sqrt{2\lambda}\right),$$

where $\mathrm{sech}(x) = 2/(e^x + e^{-x})$.

Exercises for the tutorial:
Exercises 4.4 - 4.7 will be discussed in the tutorials on November the 27th (Group 1) and some date in December yet to be decided, since December the 4th is the Dies Academicus (Groups 2 and 3).

Exercise 4.4

Let $(B_t)_{t\geq 0}$ be a Brownian Motion.

(i) For $T := \inf\{t \geq 0 : B_t = 1\}$, determine $\mathbb{E}[T]$.

(ii) For $T_{a,b} := \inf\{t \geq 0 : B_t \in \{a, b\}\}$, determine $\mathbb{E}[T_{a,b}]$.

Exercise 4.5

Prove that the maximum of a Brownian Motion $(B_t)_{t\in[0,1]}$ is almost surely attained at a unique time. Call now this time M^*. Prove that M^* is arcsine distributed, i.e. it has density

$$f_{M^*}(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \qquad \text{for } x \in [0,1].$$

Exercise 4.6

Let $(S_n)_{n\in\mathbb{N}}$ be a simple symmetric random walk on $\mathbb{Z}$. Show that

$$\#\{k \in \{1, ..., n\} : S_k > 0\} \overset{(d)}{=} \min\{k \in \{0, ..., n\} : S_k = \max_{0\leq j\leq n} S_j\}.$$

Exercise 4.7

Let $(B_t)_{t\in[0,1]}$ be Brownian Motion. Prove that $\mathcal{L}\{t \in [0,1] : B_t > 0\}$ is arcsine distributed, where $\mathcal{L}$ is the Lebesgue measure.