TU München
Zentrum Mathematik
Lehrstuhl für Wahrscheinlichkeitstheorie

Stochastic Analysis
Winter Term 2014/15
Berger/Salvi

Exercise sheet 5

Skorokhod, Donsker, Lévy, quadratic variation.

Homework:
Please hand in the solutions of Exercises from 5.1 to 5.4 in the lecture on December 11th.

Exercise 5.1 (4 Points)

Let $X \sim \mathcal{N}(0,1)$ be a standard normal random variable. Find two *different* stopping times T_1 and T_2 for a Brownian Motion $(B_t)_{t\geq 0}$ satisfying the requirements of Skorokhod's Embedding, i.e., B_{T_1} and B_{T_2} have the same distribution of X, and $\mathbb{E}[T_1] = \mathbb{E}[T_2] = Var(X)$. This shows that in general the Skorokhod's Embedding is not unique.

Exercise 5.2 (6 Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion. You will learn in the lecture that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} \left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}} \right)^2 = 1 \quad \text{in } L^2 \text{ (and a.s.).}$$

(i) Use this result to calculate the L^2-limit of

$$\sum_{i=0}^{n-1} f\left(B_{\frac{i+1}{n}}, B_{\frac{i}{n}} \right) \left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}} \right)$$

for a) $f(x,y) = x$, b) $f(x,y) = \frac{x+y}{2}$ and c) $f(x,y) = y$.

(ii) Show that the laws of $(B_t)_{t\geq 0}$ and $(3B_t)_{t\geq 0}$ are mutually singular (i.e., there exists an event A such that its probability is 1 under the law of $(B_t)_{t\geq 0}$ and is 0 under the law of $(3B_t)_{t\geq 0}$).

Exercise 5.3 (6 Points)
Take a partition $\pi = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$ of the interval $[0,1]$ and denote by $|\pi| = \max_j (t_j - t_{j-1})$ the mesh size. We say that a sequence of partitions (π_n) is *nested*, if $\pi_n \subset \pi_{n+1}$ for each n. Take any continuous function f on $[0,1]$ and define

$$S^2(\pi_n, f) = \sum_{j=1}^{n} \left(f(t_j) - f(t_{j-1}) \right)^2 .$$

(i) Let $\varepsilon > 0$. Show that there exists a partition π, such that $S^2(\pi, f) < \varepsilon$.

Please turn the page!

(ii) Show that if f is of bounded variation on $[0,1]$ and (π_n) is a nested sequence of partitions such that $|\pi_n| \to 0$, then $S^2(\pi_n, f) \to 0$.

(iii) Let $(B_t)_{0 \le t \le 1}$ be standard Brownian motion. In the lecture you learn that if the π_n's are nested and $|\pi_n| \to 0$, then $S^2(\pi_n, B) \to t$ in L^2 and a.s.. Why is this no contradiction to your previous two results?

Exercise 5.4 (4 Points)

Let $(B_t^1)_{t\ge 0}$ and $(B_t^2)_{t\ge 0}$ be two independent standard Brownian Motion and let $(X_t)_{t\ge 0}$ be the (boring) process such that $X_t = t$ for all $t \ge 0$. Calculate the cross-variations $\langle B^1, B^2\rangle_t$ and $\langle B^1, X\rangle_t$.

Exercises for the tutorial:
Exercises 5.5 - 5.7 will be discussed in the tutorials on December the 11th (Group 1) and December the 18th (Groups 2 and 3).

Exercise 5.5
Let $(B_t)_{t\ge 0}$ be a Brownian motion and $0 = t_0 < t_1 < \ldots < t_n = 1$ a partition of $[0,1]$.

(1) For $t \ge 0$, compute the stochastic integral $X_t := \int_0^t F(s)dB_s$ for the step function $F(s) := \sum_{i=0}^{n-1} f_i 1_{t_i < t \le t_{i+1}}$, where f_i are deterministic constants.

(2) Show that $(X_t)_{t\ge 0}$ is a martingale with respect to the canonical filtration $(\mathcal{F}_t)_{t\ge 0}$.

Exercise 5.6
Let $(B_t)_{t\ge 0}$ be a Brownian motion, let I_A be the indicator function on the set A and set

$$H(x) := \mathrm{I}_{\{x>0\}} - \mathrm{I}_{\{x<0\}}$$

Prove that $\int_0^1 H(B_t)dB_t$ is a normal random variable.

Exercise 5.7
Let $(B_t)_{t\ge 0}$ be a Brownian motion and $f(t)$ a square integrable function on $[0,T]$, $T > 0$.

(i) Show that

$$\int_0^T f(t)dB_t$$

is a normal random variable with mean 0 and variance $\int_0^T f(t)^2 dt$.

(ii) Give an example of a progressively measurable function $g(t,\omega)$ satisfying $E[\int_0^\infty g(t,\omega)^2 dt] < \infty$ for which

$$\int_0^\infty g(t,\omega)dB_t$$

is not a normal random variable.