Exercise sheet 6

Semi-martingales and stochastic integrals.

Homework:
Please hand in the solutions of Exercises from 6.1 to 6.2 in the lecture on January 8th.

Christmas Exercise (+1 Point):
Santa Claus has only one night to bring presents to all the children on the surface of the Earth. His magic reindeers can travel at the speed of light, but the price to pay for that is that they cannot control the direction where they go: They perform a Brownian Motion. Will Santa Claus be able to bring a present to every child on Earth with probability one? What if the surface of the Earth was infinitely large? Consider now the alien version of Santa, who has to bring a present to every alien child in the Universe, with reindeers performing a Brownian motion at infinite speed in the space. Will every alien child of the universe receive its present?

Exercise 6.1 (6 Points)

(i) Prove that a continuous martingale with a.s. bounded variation is a.s. a constant.
Hint: Use the proof of Lévy's theorem.

(ii) Prove that the decomposition for semi-martingales is unique (up to a constant and up to null sets).

(iii) Provide a process that is NOT a semi-martingale.

Exercise 6.2 (4 Points)
Let $(B_t)_{t\geq 0}$ be a Brownian motion, let I_A be the indicator function on the set A and set

$$H(x) := \mathrm{I}_{\{x>0\}} - \mathrm{I}_{\{x<0\}}$$

Let $T > 0$. Prove that $\int_0^T H(B_t)dB_t$ is a normal random variable and calculate its mean and variance.

Please turn the page!

Exercises for the tutorial:
Exercises 6.3 - 6.6 will be discussed in the tutorials on January the 8th (Group 1) and January the 15th (Groups 2 and 3).

Exercise 6.3
Let $(B_t)_{t\geq 0}$ be standard Brownian motion and consider the following stochastic integrals.

$$\text{(i)} \int_0^t s\,\mathrm{d}B_s \qquad \text{(ii)} \int_0^t B_s e^{B_s}\,\mathrm{d}B_s \qquad \text{(iii)} \int_0^t \frac{1}{1+B_s^2}\,\mathrm{d}B_s.$$

Write them as functions of the Brownian motion $(B_t)_{t\geq 0}$ without the stochastic integral.

Exercise 6.4
Let $(B_t)_{t\geq 0}$ be standard Brownian motion.

(i) Is the map $t \to \int_0^t B_s\,\mathrm{d}B_s$ almost surely continuous?

(ii) For $t \geq 0$ let $X_t = \int_0^t s\,\mathrm{d}B_s$. Is $(X_t)_{t\geq 0}$ absolutely continuous with respect to $(B_t)_{t\geq 0}$?

(iii) Is $X_t = t^2/2 + \int_0^t \sin(X_s)\,\mathrm{d}B_s$ a martingale?

Exercise 6.5
Let $(B_t)_{t\geq 0}$ be standard Brownian motion. Find an increasing process $(A_t)_{t\geq 0}$ and functions $f : \mathbb{R} \to \mathbb{R}$ such that the following processes $(M_t)_{t\geq 0}$ are continuous (but not constant) martingales with respect to their own filtration $(\mathcal{F}_t)_{t\geq 0} = \sigma(\{M_s : 0 \leq s \leq t\})_{t\geq 0}$. Let $t \geq 0$ and set

(i) $M_t = B_t^4 - A_t$

(ii) $M_t = f(t) \cdot \cos(B_t)$

(iii) $M_t = (B_t + f(t)) \cdot \exp(-B_t - t/2)$.

Exercise 6.6
Let $(B_t)_{t\geq 0}$ be standard Brownian motion.

(i) For a continuous and differentiable function $f : \mathbb{R} \to \mathbb{R}$ show that

$$M_t := f(t)B_t - \int_0^t f'(s)B_s\,\mathrm{d}s$$

is a martingale.

(ii) Compute expectation, variance and covariance function of $(M_t)_{t\geq 0}$.

(iii) Consider the random variable $T = \inf\{t \geq 0 : B_t \in \{a, b\}\}$ for $a < 0 < b$. Show that

$$\mathsf{E}[B_T \sin(T)] = \mathsf{E}\left[\int_0^T \cos(s)B_s\,\mathrm{d}s\right].$$