Exercise sheet 7

Itô formula and Girsanov.

Homework:
Please hand in the solutions of Exercises from 7.1 to 7.4 in the lecture on January 22nd.

Exercise 7.1 (8 Points)

Let $(B_t)_{t\geq 0}$ be standard Brownian motion.

(i) For a continuous and differentiable function $f : \mathbb{R} \to \mathbb{R}$ show that

$$M_t := f(t)B_t - \int_0^t f'(s)B_s \,\mathrm{d}s$$

is a martingale.

(ii) Compute expectation, variance and covariance function of $(M_t)_{t\geq 0}$.

(iii) Consider the random variable $T = \inf\{t \geq 0 : B_t \in \{a, b\}\}$ for $a < 0 < b$. Show that

$$\mathsf{E}[B_T \sin(T)] = \mathsf{E}\left[\int_0^T \cos(s)B_s \,\mathrm{d}s\right].$$

Exercise 7.2 (7 Points)

Let $(B_t)_{t\geq 0}$ be standard Brownian motion and define for $\alpha, \beta > 0$ and for all $t \geq 0$

$$M_t = \exp\left(\alpha B_t + \left(\beta - \frac{\alpha^2}{2}\right) t\right).$$

(i) Find a stochastic differential equation (i.e., an expression of the form $M_t - M_0 = \int_0^t a(M_s, s)\mathrm{d}B_s + \int_0^t b(M_s, s)\mathrm{d}s$, for some functions a and b), which is solved by M_t.

(ii) Find some value of β such that M_t is a martingale.

(iii) Calculate $\mathsf{E}[\langle M\rangle_t]$ for $t \geq 0$ and β such that M_t is a martingale.

Exercise 7.3 (5 Points)
Let $(B_t)_{t\geq 0}$ be standard Brownian motion and define $M_t := \int_0^t \exp\{B_s\}\,\mathrm{d}B_s$.

(i) Compute $\mathsf{E}[M_1 M_2]$.

Please turn the page!

(ii) Prove or disprove that

$$\mathbb{P}(M_t > 0) > 0 \quad \text{and/or} \quad \mathbb{P}(M_t < 0) > 0.$$

Exercises for the tutorial:
Exercises 7.5 - 7.7 will be discussed in the tutorials on January the 22nd (Group 1) and January the 29th (Groups 2 and 3).

Exercise 7.4

Let $(B_t)_{t\geq 0}$ be a Brownian Motion. Solve the Stochastic Differential Equation (SDE)

$$\mathrm{d}X_t = \sigma X_t \, \mathrm{d}B_t + \mu X_t \, \mathrm{d}t.$$

Exercise 7.5

Let $(B_t)_{t\geq 0}$ be a Brownian Motion. Solve the SDE

$$\mathrm{d}X_t = \theta(\mu - X_t) \, \mathrm{d}t + \sigma \, \mathrm{d}B_t.$$

Exercise 7.6

Let $(B_t)_{t\geq 0}$ be a Brownian Motion. Prove that the following SDE has a unique (strong) solution:

$$\mathrm{d}X_t = \left(\sqrt{1 + X_t^2} + \frac{X_t}{2}\right) \mathrm{d}t + \sqrt{1 + X_t^2} \, \mathrm{d}B_t.$$

Exercise 7.7
Let $(B_t)_{t\geq 0}$ be standard Brownian motion.

(i) Show that the process $(X_t)_{t\geq 0}$ defined by

$$\mathrm{d}X_t = \sin\left(\frac{1}{B_t}\right) \mathrm{d}t + \mathrm{d}B_t$$

is also Brownian motion.

(ii) Show that the stochastic differential equation

$$\begin{cases} \mathrm{d}X_t = \sin\left(\frac{1}{X_t}\right) \mathrm{d}t + \mathrm{d}B_t \\ X_0 = x_0 \end{cases}$$

has a weak solution on every interval $[0, T]$, $T \geq 0$.