Exercise sheet 8

Recap.

Read me:

- There is no homework this time! The exercises (probably only part of them) will be discussed during the Q&A session on the 12th of February, at 10h30, in the usual main lecture room (I apologize with those who have an exam on the same day). Try to solve the exercises by yourself beforehand, so that you will have the right questions to ask when we meet. Exercises with a star are taken from last years' exams. We do NOT plan to publish the solutions of these exercises online (sorry).
- Note that the following exercises are more focused on the second part of the course, but the exam will be on the entire program, with topics chosen more or less evenly along the semester (read: You will not be asked only Itô's formula). Remember that during the exam you will be allowed to consult only your brain: No books or notes will be admitted!

*Exercise 8.1

Fix $1 < T < \infty$. Decide whether each of the following is a well defined Itô integral or not. Justify your answer.

$$\text{(i)} \int_{1/2}^{1} B_{\frac{1}{t}} \mathrm{d}B_t \qquad \text{(ii)} \int_{1}^{T} B_{\frac{1}{t}} \mathrm{d}B_t \qquad \text{(iii)} \int_{1}^{\infty} B_{\frac{1}{t}} \mathrm{d}B_t.$$

*Exercise 8.2

Let $(B_t)_{t\in[0,1]}$ be a Brownian Motion on the interval $[0, 1]$, and let P be its distribution.

(i) For $0 \leq t \leq 1$, let $X_t = B_t + t$, and let Q be the distribution of (X_t). Prove or disprove that $Q \ll P$.

(ii) For $0 \leq t \leq 1$, let $X_t = B_{t/2}$, and let Q be the distribution of (X_t). Prove or disprove that $Q \ll P$.

Reminder: A measure μ on a given σ-algebra is absolutely continuous with respect to a second measure λ if, for every measurable set A, $\lambda(A) = 0 \Rightarrow \mu(A) = 0$. We write $\mu \ll \lambda$.

*Exercise 8.3

Show that there exists an infinite sequence of independent random variables, which are $\mathcal{N}(0, 1)$-distributed (normally distributed with mean zero and variance one) and measurable with respect to $\mathcal{F}_1 := \sigma(B_t : 0 \leq t \leq 1)$.

Please turn the page!

***Exercise 8.4**
Show that, for $T < \infty$, there exists a weak solution $(Y_t)_{t\in[0,T]}$ of

$$\begin{cases} \mathrm{d}Y_t = \mathrm{d}B_t + \sqrt{Y_t}\mathrm{d}t & \text{for all } t \in [0,T] \\ Y_0 = 0. \end{cases}$$

Hint: remember that $\int_0^t B_s \mathrm{d}s \sim \mathcal{N}\left(0, \frac{t^3}{3}\right)$.

***Exercise 8.5**
Let $X = \max\{B_t : 0 \leq t \leq 1\}$ and set $Y := B_2 - B_1$.

(i) Prove that $P(X > Y) \geq 1/2$.

(ii) Calculate the distribution functions of X and of $Z := |Y|$.

(iii) Prove or disprove that $P(X > Z) = 1/2$.

Exercise 8.6
Let $(B_t)_{t\geq 0}$ be standard Brownian motion. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that the processes $(M_t)_{t\geq 0}$ given by $M_t = (B_t + f(t)) \cdot \exp(-B_t - t/2)$ is a continuous (but not constant) martingale with respect to their own filtration $(\mathcal{F}_t)_{t\geq 0} = \sigma(\{M_s : 0 \leq s \leq t\})_{t\geq 0}$.

Exercise 8.7
Consider two independent Brownian motions $(B_t)_{t\geq 0}$ and $(C_t)_{t\geq 0}$ on some complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

(i) Determine the subset $\Lambda \subset \mathbb{R}^2$ of ***all(!)*** parameters $(\alpha, \beta) \in \mathbb{R}^2$ such that the process $(D_t)_{t\geq 0}$ where

$$D_t := \alpha B_t + \beta C_t \qquad \text{for } t \geq 0$$

is again a Brownian motion and give reasons for your choice for Λ.

(ii) Determine a probability measure $\widetilde{\mathbb{P}}$ on $(\Omega, \mathcal{A})$ such that the process $(G_t)_{0\leq t\leq 1}$ where

$$G_t := B_t + t \qquad \text{for } 0 \leq t \leq 1$$

is a Brownian motion with respect to $\widetilde{P}$. Further compute

$$\mathsf{E}_{\widetilde{\mathbb{P}}}[(B_t)^2]$$

for $0 \leq t \leq 1$ where $\mathsf{E}_{\widetilde{\mathbb{P}}}[\cdot]$ denotes the expectation with respect to $\widetilde{\mathbb{P}}$.

Exercise 8.8
Let $(B_t)_{t\geq 0}$ be a Brownian Motion. Solve the SDE

$$\mathrm{d}X_t = \theta(\mu - X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t.$$