Solutions for Exercise sheet 5

Skorokhod, Donsker, Lévy, quadratic variation.

Solution for Exercise 5.1

The easiest choice for T_1 is undoubtly the deterministic time $T_1 = 1$. Clearly $B_{T_1} = B_1 \sim X$ and $\mathbb{E}[T_1] = 1 = Var(X)$. We have to find a $T_2 \neq T_1$ with the same characteristics. The proof of Skorokhod Embedding Theorem is constructive, and provide an explicit description of a stopping time with the required properties.

Consider $v_1^1 := \mathbb{E}[X|X > 0]$ and $v_2^1 := \mathbb{E}[X|X < 0]$; call S_1 the first time that $B_t \in \{v_1, v_2\}$. If $B_{S_1} = v_1^1$, then call S_2 the first time that $B_t \in \{v_1^2, v_2^2\}$, where $v_1^2 := \mathbb{E}[X|X > 0,\, X > v_1^1]$ and $v_2^2 := \mathbb{E}[X|X > 0,\, X < v_1^1]$. If instead $B_{S_1} = v_2^1$, then define v_3^2 and v_4^2 in a symmetric way and S_2 analogously (the best thing you can do is drawing a picture of all of this). Continue with this procedure infinitely many times, and obtain a sequence $S_1 < S_2 < ...$ of stopping times that converge a.s. to a stopping time T (our T_2). The proof of Skorokhod Embedding guarantees that $\mathbb{E}[T_2] = Var(X) = 1$ and that $B_{T_2} \sim X$.

We are left to show that $T_1 \neq T_2$. But this is clear, since T_2 is random, while T_1 is deterministic. In particular, $T_2 \geq S_1$, and S_1 can be bigger than 1.

Solution for Exercise 5.2

(i) (a) First convince yourself that

$$B_{\frac{i+1}{n}}\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right) = \frac{1}{2}\left(B^2_{\frac{i+1}{n}} - B^2_{\frac{i}{n}}\right) + \frac{1}{2}\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right)^2.$$

Then get the desired result by taking the limit in the L^2 sense (the first term is just a telescopic sum!),

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=0}^{n-1} B_{\frac{i+1}{n}}\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right) &= \frac{1}{2}B_1^2 + \frac{1}{2}\lim_{n\to\infty}\sum_{i=0}^{n-1}\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right)^2 \\ &= \frac{1}{2}(B_1^2 + 1).\end{aligned}$$

(c) In this case we just have to modify the previous result by using

$$B_{\frac{i}{n}}\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right) = \frac{1}{2}\left(B^2_{\frac{i+1}{n}} - B^2_{\frac{i}{n}}\right) - \frac{1}{2}\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right)^2$$

instead. This gives us

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} B_{\frac{i}{n}}\left(B_{\frac{i+1}{n}} - B_{\frac{i}{n}}\right) = \frac{1}{2}(B_1^2 - 1).$$

(b) A direct calculation or a linear combination of the previous two results gives

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\frac{B_{\frac{i+1}{n}}+B_{\frac{i}{n}}}{2}\left(B_{\frac{i+1}{n}}-B_{\frac{i}{n}}\right)=\frac{1}{2}B_1^2.$$

In this exercise you have calculated three versions of the stochastic integral $\int_0^1 B_s \mathrm{d}B_s$. The integral in part c) is the so called Itô Integral that you will use in the rest of the course. In part b) you calculated the Stratonovich integral. We see that the approximation of the stochastic integral by step functions depends on the choice of the approximation f. Note that only the approximation given in c) is a progressively measurable step function! In contrast, the Riemann integral does not depend on the choice of f!

(ii) Using the notation from Exercise 5.3 we let π_n be a nested sequence of partitions of $[0,1]$ with $|\pi_n|\to 0$. Then the quadratic variation of standard Brownian motion (B_t) is

$$\lim_{n\to\infty} S^2(\pi_n, B)=1.$$

We also see that for the scaled Brownian motion $(3B_t)$ the quadratic variation is

$$\lim_{n\to\infty} S^2(\pi_n, 3B)=9.$$

Thus, define the set

$$A:=\left\{f\in C[0,1]: \text{the quadratic variation of } f \text{ exists and } \lim_{n\to\infty} S^2(\pi_n,f)=1\right\}.$$

Then $\mathsf{P}((B_t)_{0\le t\le 1}\in A)=1$ but $\mathsf{P}((3B_t)_{0\le t\le 1}\in A)=0$ and consequently the laws of both processes are mutually singular.

Solution for Exercise 5.3

(i) We must find a partition *ad hoc* exploiting the shape of f. Without loss of generality, suppose $f(1)>f(0)$ (the converse case can be handled exactly in the same way, while the case $f(1)=f(0)$ is trivial). Let $I=I(f)$ be the interval $I=[f(0),f(1)]$ and chop it in tiny intervals $I_1,...,I_n$ of length $\frac{1}{n}$, such that

$$I_j=[\tfrac{(f(1)-f(0))(j-1)}{n}+f(0), \tfrac{(f(1)-f(0))j}{n}+f(0)]=[i_j,i_{j+1}].$$

Define the points of our partition $\pi_n=\{0=t_1<...<t_n<t_{n+1}=1\}$ as follows: $t_j=\min\{t\in[0,1]:\ f(t)=i_j\}$.

Then

$$\begin{aligned}S^2(\pi_n,f)&=\sum_{j=1}^{n}(f(t_{j+1})-f(t_j))^2\\&=\sum_{j=1}^{n}\tfrac{1}{n}^2\\&=\tfrac{1}{n}.\end{aligned}$$

Taking n big enough gives the claim. Were did we use continuity?

REMARK: A picture will help you more than any tutor.

(ii) Denote with $TV(f)$ the variation of f (i.e., the supremum over all possible partitions of $\sum_{j=1}^{n-1} |f(t_{j+1}) - f(t_j)|$).

$$\begin{aligned} S^2(\pi_n, f) &= \sum_{j=1}^{n-1} (f(t_{j+1}) - f(t_j))^2 \\ &\le \max_j |f(t_{j+1}) - f(t_j)| \sum_{j=1}^{n-1} |f(t_{j+1}) - f(t_j)| \\ &\le \max_j |f(t_{j+1}) - f(t_j)| TV(f) \end{aligned}$$

Since the mesh size of the partitions is going to 0 and f is continuous, we have $S^2(\pi_n, f) \to 0$.

(iii) In part (i) the mesh size of the partition(s) we chose was not going to 0. However, we could modify the proof such that the results holds true even if we require $|\pi_n| \to 0$. But we still would have no contradiction, because the partition π_n is dependent on the sample path $B_t(\omega)$, while $S^2(\pi_n, B) \to t$ only if π_n is independent of $B_t(\omega)$. In part (ii) we took f to have bounded variation, while we know that the Brownian Motion has almost surely *unbounded* variation. We are safe.

Solution for Exercise 5.4

- Take any refining sequence of partitions π_n of the interval $[0, t]$. Call its length ℓ_n and call its elements $\pi_n(k)$ for $k = 1, ..., \ell_n$. Then

$$\begin{aligned} \langle B^1, X \rangle_t &= \lim_{n\to\infty} \sum_{k=1}^{\ell_n} (B^1_{\pi_n(k)} - B^1_{\pi_n(k-1)})(\pi_n(k) - \pi_n(k-1)) \\ &\le \lim_{n\to\infty} \Big(\sum_{k=1}^{\ell_n} (B^1_{\pi_n(k)} - B^1_{\pi_n(k-1)})^2 \Big)^{1/2} \Big(\sum_{k=1}^{\ell_n} (\pi_n(k) - \pi_n(k-1))^2 \Big)^{1/2} \\ &= t \cdot \lim_{n\to\infty} \Big(\sum_{k=1}^{\ell_n} (\pi_n(k) - \pi_n(k-1))^2 \Big)^{1/2} \\ &= 0 \end{aligned}$$

where we have used Cauchy-Schwarz inequality for the second line and the fact that $(X_s)_{s\in[0,t]}$ has 0 quadratic variation (why?) for the last line.

- We take now the other definition of cross variation: We know that

$$\langle B^1, B^2 \rangle_t = 4\big(\langle B^1 + B^2 \rangle_t - \langle B^1 - B^2 \rangle_t\big) \tag{1}$$

But now observe that, since $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ are independent, the processes $\left(\frac{1}{\sqrt{2}}(B_t^1 + B_t^2)\right)_{t\geq 0}$ and $\left(\frac{1}{\sqrt{2}}(B_t^1 - B_t^2)\right)_{t\geq 0}$ are both standard Brownian Motions (it takes 2 lines to prove it, if you do not believe it)! Therefore the two quadratic variations in the right hand side of equation (1) are exactly the same (equal to $2t$) and cancel out.

Therefore, a.s.,

$$\langle B^1, B^2 \rangle_t = 0.$$