Solutions for Exercise sheet 6

Semi-martingales and stochastic integrals.

Solution for Christmas Exercise

Since the surface of Earth is two-dimensional, even if it was infinitely big, Santa would manage to get arbitrarily close to every point on the surface (check *recurrence*), i.e. arbitrarily close to any child in the world, and would be able to hand in his amazing presents.

He would have a much harder time in the universe: in three dimensions his magic random reindeers would get lost in space and would never come back to their origin (check *transience*). As a consequence, he would not be able to get arbitrarily close to every alien child.

Solution for Exercise 6.1

(i) Let τ be the time parametrization of the martingale in Lévy's representation theorem and α its inverse, as in the lecture notes. The theorem tells us that $(M(\alpha(t)))_{t\geq 0}$ is a Brownian Motion. By the construction of τ we can see that $\alpha(t)$ is almost surely finite if the martingale is almost surely unbounded.

Take some $n \in \mathbb{N}$ (if the martingale is bounded between two values, say $A < 0$ and $B > 0$, and is not constant, take $n \in \mathbb{N}$ big enough so that $-2^{-n} > A$ and $2^{-n} < B$). The martingale will almost surely touch either -2^{-n} or 2^{-n} in finite time, call this time $T^M_{\pm 2^{-n}}$. Through the construction of τ, we see that $T^M_{\pm 2^{-n}}$ is mapped to $T^B_{\pm 2^{-n}} > 0$, the first time the Brownian Motion touches respectively -2^{-n} or 2^{-n}, and the path of the Martingale up to time $T^M_{\pm 2^{-n}}$ is just a reparametrization of that of the Brownian Motion up to time $T^B_{\pm 2^{-n}}$: Therefore it cannot have bounded variation!

Note that if the martingale was a.s. constant, the whole path of the martingale would have been mapped to the point B_0.

(ii) Let $X(t)$ be our semi-martingale and suppose that there exist two different decompositions: $X(t) = M(t)+F(t)$ and $X(t) = M'(t)+F'(t)$, with $M(t)$ and $M'(t)$ two continuous martingales and $F(t)$ and $F'(t)$ two continuous (adapted) functions of bounded variation. Obviously

$$M(t) + F(t) - M'(t) - F'(t) = 0 \qquad a.s.\ \forall t \geq 0. \tag{1}$$

The process $(M(t) - M'(t))_{t\geq 0}$ is a martingale (because it is the difference of two martingales). Suppose it is not constant. Then, by part (i), it must have infinite variation. By (1), also $(F'(t) - F(t))_{t\geq 0}$ must have infinite variation. But this is not possible, since both F and F' have finite variation.

(iii) It is enough to find a deterministic process that is not a semi-martingale. Any continuous function that has infinite variation will do the job (it clearly cannot be decomposed into a martingale + a function of finite variation): For example one could choose a particular path of the Brownian Motion $(B_t(\omega))_{t\in[0,1]}$ (with ω fixed!).

Solution for Exercise 6.2

$(H(B_t))_{t\geq 0}$ is progressively measurable, since it is the composition of a $\mathcal{B}([0,t])\times\mathcal{F}_t$-measurable map (the Brownian motion) with another measurable map. Also we easily see that the process is in L^2:

$$\mathsf{E}\Big[\int_0^T H^2(B_t)\,\mathrm{d}t\Big] \leq \int_0^T 1\,\mathrm{d}t = T < \infty.$$

As usual, let (π_n) ba a sequence of partitions of $[0,T]$ with mesh-size tending to 0 ($|\pi_n|\to 0$). We defined the stochastic integral as the following (L^2-)limit:

$$\int_0^T H(B_t)\,\mathrm{d}B_t := \lim_{n\to\infty}\sum_{j=1}^{N_n} H(B_{t_j})(B_{t_{j+1}}-B_{t_j}), \tag{2}$$

where $\pi_n = \{0 = t_0 < t_1 < ... < t_{N_n} = T\}$.

Let us look at a particular j and call $X := H(B_{t_j})$ and $Y := B_{t_{j+1}} - B_{t_j}$ (notice that $Y \sim \mathcal{N}(0, t_{j+1}-t_j)$. We need to understand the distribution of $X\cdot Y$. For any value $c\in\mathbb{R}$:

$$\begin{aligned} P(XY\leq c) &= \sum_{k=0,1,-1} P(X=k,\, kY\leq c)\\ &= \sum_{k=0,1,-1} P(X=k)\,P(kY\leq c)\\ &= \frac{1}{2}P(Y\leq c) + \frac{1}{2}P(-Y\leq c)\\ &= P(Y\leq c) \end{aligned} \tag{3}$$

where in the second line we used the independence of X and Y (why are they independent?), in the third line the fact that BM is equal to 0 with probability 0, and in the fourth line the fact that Y is symmetric. Therefore, every summand of (2) is normally distributed, with mean 0 and variance, respectively, $t_{j+1}-t_j$. The other important fact that is left to notice is that each summand is independent of the others (how is this possible?). Being the sum of independent Gaussians, the sum is distributed as a Gaussian random variable with mean 0 (sum of the means) and variance T (sum of the variances).