Solutions for Exercise sheet 7

Itô formula and Girsanov.

Solution for Exercise 7.1

(i) Set $g(x,t) := xf(t)$. We would like to apply Itô formula to g.

- $\mathbb{E}\left[\int_0^t f^2(s)B_s^2\,\mathrm{d}s\right] = \int_0^t f^2(s)s\,\mathrm{d}s$ by Fubini Tonelli, and we see that the integral is finite because f, being continuously differentiable, has to be bounded in the compact interval $[0,t]$, for all $t \geq 0$.
- g is clearly twice continuously differentiable in x and is once continuously differentiable in t by hypothesis.

We can therefore apply Itô and we get

$$f(t)B_t = \int_0^t f(s)\,\mathrm{d}B_s + \int_0^t B_s f'(s)\,\mathrm{d}s + 0.$$

Rearranging

$$M_t = \int_0^t f(s)\mathrm{d}B_s,$$

which is a martingale (it is the stochastic integral of a progressively measurable function).

(ii)
- $(M_t)_{t\geq 0}$ is a martingale $\Longrightarrow \mathbb{E}[M_t] = \mathbb{E}[M_0] = 0$.
- $Var(M_t) = \mathbb{E}[M_t^2] = \int_0^t f(s)^2\,\mathrm{d}s$, where for the second equality we have used Itô isometry.
- For $s < t$,

$$\begin{aligned} Cov(M_s, M_t) &= \mathbb{E}[(M_s - \mathbb{E}[M_s])(M_t - \mathbb{E}[M_t])] \\ &= \mathbb{E}[M_s \mathbb{E}[M_t|\mathcal{F}_s]] \\ &= \mathbb{E}[M_s^2] \\ &= \int_0^s f(u)^2\,\mathrm{d}u, \end{aligned} \tag{1}$$

where we have used the tower property of conditional expectation and the measurability of M_s w.r.t. $\mathcal{F}_s$ in the second line, the martingale property in the third and Itô isometry for the last line.

(iii) By (i) with $f(t) = \sin(t)$, we know that $M_t := B_t \sin(t) - \int_0^t B_s cos(s)\,\mathrm{d}s$ is a martingale. We would like to use the Optional Stopping Theorem for proving the claim. Notice that

$$\begin{aligned}|M_{t\wedge T}| &\leq |B_{t\wedge T}| + |\int_0^{t\wedge T} B_s \cos(s)\,\mathrm{d}s| \\ &\leq (|a| \vee b) + \int_0^{t\wedge T} |B_s|\,\mathrm{d}s \\ &\leq (|a| \vee b) + (|a| \vee b)T =: Y.\end{aligned}$$

Since $\mathbb{E}[T] = |a| \cdot b$ (see Exercise 4.4 (ii)), the random variable Y is such that $\mathbb{E}[|Y|] < \infty$ and therefore we can apply the OST. This tells us that, in particular,

$$\mathbb{E}[M_T] = \mathbb{E}[M_0] = 0.$$

Rearranging the summands we obtain the claim.

Solution for Exercise 7.2

(i) We want to apply Itô to the function $f(x,t) = \mathrm{e}^{\alpha x + (\beta - \alpha^2/2)t}$. Note that

$$\begin{aligned}\partial_x f(x,t) &= \alpha f(x,t) \\ \partial_{xx} f(x,t) &= \alpha^2 f(x,t) \\ \partial_t f(x,t) &= (\beta - \tfrac{\alpha^2}{2}) f(x,t)\end{aligned}$$

are all continuous. Furthermore

$$\begin{aligned}\mathbb{E}\Big[\int_0^t (\partial_x f(B_s,s))^2\,\mathrm{d}s\Big] &= \mathbb{E}\Big[\int_0^t \alpha^2 f(B_s,s)^2\,\mathrm{d}s\Big] \\ &= \alpha^2 \int_0^t E\Big[f(B_s,s)^2\Big]\,\mathrm{d}s \\ &= \alpha^2 \int_0^t E\Big[\mathrm{e}^{2\alpha B_s}\Big]\mathrm{e}^{(\beta-\alpha^2/2)s}\,\mathrm{d}s \\ &= \alpha^2 \int_0^t \mathrm{e}^{\frac{4\alpha^2}{2}s}\mathrm{e}^{(\beta-\alpha^2/2)s}\,\mathrm{d}s < \infty,\end{aligned}$$

where we have used Fubini-Tonelli in the second line (the integrand was positive) and Exercise 2.5 for the last line. We are therefore allowed to use Itô formula! This gives

$$\begin{aligned}f(B_t,t) - 1 &= \int_0^t \alpha f(B_s,s)\,\mathrm{d}B_s + \int_0^t (\beta - \tfrac{\alpha^2}{2}) f(B_s,s)\,\mathrm{d}s + \frac{1}{2}\int_0^t \alpha^2 f(B_s,s)\,\mathrm{d}s \\ &= \int_0^t \alpha f(B_s,s)\,\mathrm{d}B_s + \int_0^t \beta f(B_s,s)\,\mathrm{d}s.\end{aligned}$$

(ii) Form the last display, we see that $M_t = \int_0^t \alpha f(B_s, s)\,\mathrm{d}B_s$ if $\beta = 0$. In that case M_t is a martingale, being a stochastic integral.

(iii)

$$\mathbb{E}[\langle M\rangle_t] = \mathbb{E}\Big[\int_0^t (\alpha f(B_s, s))^2 \mathrm{d}s\Big],$$

where we have used a Proposition from the last exercise class for the calculation of quadratic variation of stochastic integrals. Proceed with the calculation as in (i).

Solution for Exercise 7.3

(i) First note that the stochastic integral is well-defined since by Fubini-Tonelli

$$\mathsf{E}\left[\int_0^t e^{2B_s}\mathrm{d}s\right] = \int_0^t \mathsf{E}[e^{2B_s}]\mathrm{d}s = \int_0^t e^{2s}\mathrm{d}s = (e^{2t}-1)/2 < \infty.$$

M_t is a martingale, since it is a stochastic integral. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of Brownian motion. Then

$$\begin{aligned}\mathsf{E}[M_1M_2] &= \mathsf{E}[\mathsf{E}[M_1M_2|\mathcal{F}_1]]\\ &= \mathsf{E}[M_1\mathsf{E}[M_2|\mathcal{F}_1]]\\ &= \mathsf{E}[M_1^2]\\ &= (e^2-1)/2.\end{aligned}$$

(ii) For the case $t = 0$ we see immediately that $\mathsf{P}(M_0 > 0) = 0$. From now on assume that $t > 0$. Since M_t is a martingale, we know that $\mathsf{E}[M_t] = \mathsf{E}[M_0] = 0$, but $\mathrm{Var}\,(M_t) = \mathsf{E}[M_t^2] = (e^{2t}-1)/2 > 0$, so for all $t > 0$ the random variable M_t has centred distribution with probability mass above and below zero, i.e. $\mathsf{P}(M_t \neq 0) > 0$. Now, w.l.o.g. assume $\mathsf{P}(M_t > 0) > 0$. Then also $\mathsf{P}(M_t < 0) > 0$, since otherwise $\mathsf{E}[M_t] > 0$. Therefore, for $t > 0$ we have found that $\mathsf{P}(M_t > 0) > 0$ and $\mathsf{P}(M_t < 0) > 0$.