Solutions for Exercise sheet 1

Multivariate Gaussian Random Variables

Solution for Exercise 1.1

(i) We calculate the characteristic function of $A\vec{X}+b$. Use $\langle \cdot, \cdot \rangle$ to denote the Euclidean scalar product. For any $\vec{t} \in \mathbb{R}^k$

$$\begin{aligned}
\varphi_{A\vec{X}+\vec{b}}(\vec{t}) &= E\left[\exp\left(i\langle \vec{t}, A\vec{X}+\vec{b}\rangle\right)\right] \\
&= \exp\left(i\langle \vec{t}, \vec{b}\rangle\right) E\left[\exp\left(i\langle A^{tr}\vec{t}, \vec{X}\rangle\right)\right] \\
&\overset{\text{Hint}}{=} \exp\left(i\langle \vec{t}, \vec{b}\rangle\right) \exp\left(i\langle A^{tr}\vec{t}, \vec{\mu}\rangle - \frac{1}{2}\langle A^{tr}\vec{t}, \Sigma(A^{tr}\vec{t})\rangle\right) \\
&= \exp\left(i\langle \vec{t}, \vec{b}+A\vec{\mu}\rangle\right) \exp\left(-\frac{1}{2}\langle \vec{t}, (A\Sigma A^{tr})\vec{t}\rangle\right).
\end{aligned}$$

This is the characteristic funtion of a $\mathcal{N}(\vec{b}+A\vec{\mu}, A\Sigma A^{tr})$ random variable and the distribution of a random variable is determined uniquely by its characteristic function.

(ii) We rewrite the quadratic form in matrix notation

$$f(x_1, x_2) = \frac{\sqrt{7}}{2\pi} \exp\left(-\frac{1}{2}\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right).$$

Thus we can read off the inverse Σ^{-1} of the covariance matrix and get

$$\Sigma = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{7}\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$$

with determinant $\det(\Sigma) = 1/7$. We see that $f(x_1, x_2)$ is the density of a bivariate random variable with mean $\vec{\mu} = 0$ and covariance matrix Σ. In particular the marginals are $X_1 \overset{d}{=} \mathcal{N}(0, 2/7) X_2 \overset{d}{=} \mathcal{N}(0, 4/7)$. Since $\mathbb{C}\mathrm{ov}(X_1, X_2) = -\frac{1}{7} \neq 0$ we conclude that X_1 and X_2 are not independent.

Solution for Exercise 1.2

We set $f(k) := E[X^k]$ and compute $f(k)$ for all $k \geq 2$ using integration by parts,

$$\begin{aligned}
f(k) &= \int_{-\infty}^{\infty} x^{k-1} \frac{x}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} (k-1)x^{k-2} \frac{\sigma^2}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= (k-1)\sigma^2 f(k-2).
\end{aligned}$$

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Note that $f(0) = 1$ and $f(1) = 0$. By recursion we get

$$f(k) = \begin{cases} 0 & \text{for } k \text{ odd} \\ \sigma^k (k-1)(k-3)\dots 1 & \text{for } k \text{ even.} \end{cases}$$